

The characterizations and representations for the generalized inverses with prescribed idempotents in Banach algebras

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Abstract

In this paper, we investigate the various different generalized inverses in a Banach algebra with respect to prescribed two idempotents p and q . Some new characterizations and explicit representations for these generalized inverses, such as $a_{p,q}^{(2)}$, $a_{p,q}^{(1,2)}$ and $a_{p,q}^{(2,l)}$ will be presented. The obtained results extend and generalize some well-known results for matrices or operators.

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1 Introduction and preliminaries

Let X be a Banach space and $B(X)$ be the Banach algebra of all bounded linear operators on X . For $A \in B(X)$, let T and S be closed subspaces of X . Recall that the out inverse $A_{T,S}^{(2)}$ with prescribed the range T and the kernel S is the unique operator $G \in B(X)$ satisfying $GAG = G$, $\text{Ran}(G) = T$, $\text{Ker}(G) = S$. It is well known that $A_{T,S}^{(2)}$ exists if and only if

$$\text{Ker}(A) \cap T = \{0\}, \quad AT \dot{+} S = Y. \quad (1.1)$$

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This type of general inverse was generalized to the case of rings by Djordjević and Wei in [4]. Let \mathcal{R} be a unital ring and let \mathcal{R}^\bullet denote the set of all idempotent elements in \mathcal{R} . Given $p, q \in \mathcal{R}^\bullet$. Recall that an element $a \in \mathcal{R}$ has the (p, q) -outer generalized inverse $b = a_{p,q}^{(2)} \in \mathcal{R}$ if

$$bab = b, \quad ba = p, \quad 1 - ab = q.$$

If $b = a_{p,q}^{(2)}$ also satisfies the equation $aba = a$, then we say $a \in \mathcal{R}$ has the (p, q) -generalized inverse b , in this case, written $b = a_{p,q}^{(1,2)}$. If an outer generalized inverse with prescribed idempotents exists, it is necessarily unique [4]. According to this definition, many generalized inverses such as the Moore–Penrose inverses in a C^* -algebra and (generalized) Drazin inverses in a Banach algebra can be expressed by some (p, q) -outer generalized inverses [4, 5].

Now we give some notations in this paper. Throughout this paper, \mathcal{A} is always a complex Banach algebra with the unit 1. Let $a \in \mathcal{A}$. If there exists $b \in \mathcal{A}$ such that $aba = a$, then a is called inner regular and b is called an inner generalized inverse (or $\{1\}$ inverse) of a , denoted by $b = a^-$. If there is an element $b \in \mathcal{A}$ such that $bab = b$, then b is called an outer generalized inverse (or $\{2\}$ inverse) of a . We say that b is a (reflexive) generalized inverse (or $\{1, 2\}$ inverse) of a , if b is both an inner and an outer generalized inverse of a (certainly such an element b is not unique). In this case, we let a^+ denote one of the generalized inverses of a . Let $Gi(\mathcal{A})$ denote the set of a in \mathcal{A} such that a^+ exists. It is well-known that if b is an inner generalized inverse of a , then bab is a generalized inverse of a (cf. [1, 11, 16]).

Let \mathcal{A}^\bullet denote the set of all idempotent elements in \mathcal{A} . If $a \in Gi(\mathcal{A})$, then a^+a and $1 - aa^+$ are all idempotent elements. For $a \in \mathcal{A}$, set

$$\begin{aligned} K_r(a) &= \{x \in \mathcal{A} \mid ax = 0\}, & R_r(a) &= \{ax \mid x \in \mathcal{A}\}; \\ K_l(a) &= \{x \in \mathcal{A} \mid xa = 0\}, & R_l(a) &= \{xa \mid x \in \mathcal{A}\}. \end{aligned}$$

Clearly, if $p \in \mathcal{A}^\bullet$, then \mathcal{A} has the direct sum decompositions:

$$\mathcal{A} = K_r(p) \dot{+} R_r(p) \quad \text{or} \quad \mathcal{A} = K_l(p) \dot{+} R_l(p).$$

In this paper, we give a new definition of the generalized inverse with prescribed idempotents and discuss the existences of various different generalized inverses with prescribed idempotents in a Banach algebra. We also give some new characterizations and explicit representations for these generalized inverses. Also, we corrected Theorem 1.4 from [5].

2 Some existence conditions for the (p, q) -outer generalized inverse

Theorem 1.4 of [5] gives three equivalent characterizations of the (p, q) -outer generalized inverse $a_{p,q}^{(2)}$ which say:

Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$. Then the following statements are equivalent:

- (i) $a_{p,q}^{(2)}$ exists;
- (ii) There exists some element $b \in \mathcal{A}$ satisfying

$$bab = b, \quad R_r(b) = R_r(p), \quad K_r(b) = R_r(q);$$

- (iii) There exists some element $b \in \mathcal{A}$ satisfying

$$bab = b, \quad b = pb, \quad p = bap, \quad b(1 - q) = b, \quad 1 - q = (1 - q)ab.$$

We see that $\mathbf{bab} = \mathbf{b}$ is redundant in Statement (iii). In fact, by using some other equations in (iii), we can check that $bab = bapb = pb = b$. Also from the definition of $a_{p,q}^{(2)}$, it is easy to check that if $a_{p,q}^{(2)}$ exists, then Statements (ii) and (iii) hold. But we can show by following example that Statement (iii) does not imply Statement (i).

Example 2.1. Consider the matrix algebra $\mathcal{A} = M_2(\mathbb{C})$, Let

$$a = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad p = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad 1_2 - q = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It is obvious that p and $1_2 - q$ are idempotents. Moreover, we have

$$pb = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = b, \quad bap = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = p,$$

$$b(1_2 - q) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = b, \quad (1_2 - q)ab = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 1_2 - q,$$

i.e., a, b, p, q satisfy Statement (iii) of Theorem 1.4 in [5], where 1_2 is the unit of \mathcal{A} . But

$$ba = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq p, \quad ab = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq 1_2 - q.$$

Therefore, by the uniqueness of $a_{p,q}^{(2)}$, we see $a_{p,q}^{(2)}$ does not exist.

By using the following auxiliary lemma, without equation $bab = b$, we prove that statements (ii) and (iii) in [5, Theorem 1.4] are equivalent, and we can also prove that the b is unique if it exists.

Lemma 2.2. *Let $x \in \mathcal{A}$ and $p \in \mathcal{A}^\bullet$. Then*

- (1) $K_r(p)$ and $R_r(p)$ are all closed and $K_r(p) = R_r(1 - p)$, $R_r(p)\mathcal{A} \subset R_r(p)$;
- (2) $px = x$ if and only if $R_r(x) \subset R_r(p)$ or $K_l(p) \subset K_l(x)$;
- (3) $xp = x$ if and only if $K_r(p) \subset K_r(x)$ or $R_l(x) \subset R_l(p)$.

Proposition 2.3. *Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$. Then the following statements are equivalent:*

- (1) *There is $b \in \mathcal{A}$ satisfying $bab = b$, $R_r(b) = R_r(p)$ and $K_r(b) = R_r(q)$;*
- (2) *There is $b \in \mathcal{A}$ satisfying $b = pb$, $p = bap$, $b(1 - q) = b$, $1 - q = (1 - q)ab$.*

If there exists some b satisfying (1) or (2), then it is unique.

Proof. (1) \Rightarrow (2) From $bab = b$, we know that $ba, ab \in \mathcal{A}^\bullet$. Then by using $R_r(b) = R_r(p)$ and $K_r(b) = R_r(q)$, we get

$$R_r(ba) = R_r(b) = R_r(p), \quad K_r(ab) = K_r(b) = R_r(q) = K_r(1 - q).$$

Since $ba, ab \in \mathcal{A}^\bullet$, then by Lemma 2.2, we have

$$b = pb, \quad p = bap, \quad b(1 - q) = b, \quad 1 - q = (1 - q)ab.$$

(2) \Rightarrow (1) We have already known that $bab = bapb = pb = b$.

From $pb = b$ we get $R_r(b) \subset R_r(p)$. Since $p = bap$ and $bab = b$, we get $R_r(p) \subset R_r(ba) = R_r(b)$. Thus $R_r(b) = R_r(p)$.

Similarly, From $b(1 - q) = b$, $1 - q = (1 - q)ab$, we can check that $K_r(b) = R_r(q)$ by using Lemma 2.2.

Now we show b is unique if it exists. In fact, if there exist b_1 and b , then

$$b_1 = pb_1 = bapb_1 = b(1 - q)ab_1(1 - q) = b(1 - q) = b.$$

That is, b is unique. □

By means of Lemma 2.2 and Proposition 2.3, we can give a correct version of [5, Theorem 1.4] as follows.

Theorem 2.4. *Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$. Then the following statements are equivalent:*

- (1) *There exists $b \in \mathcal{A}$ satisfying $bab = b$, $ba = p$ and $1 - ab = q$.*
- (2) *There exists $b \in \mathcal{A}$ satisfying*

$$bab = b \quad \text{and} \quad \begin{cases} R_r(ba) = R_r(p), & K_r(ab) = R_r(q) \\ R_l(ba) = R_l(p), & K_l(ab) = R_l(q). \end{cases}$$

Proof. The implication (1) \Rightarrow (2) is obvious. We now prove (2) \Rightarrow (1).

Suppose that (2) holds. Then $bab = b$ is a known equivalent condition. By Proposition 2.3, we have

$$b = pb, \quad p = bap, \quad b(1 - q) = b, \quad 1 - q = (1 - q)ab. \quad (2.1)$$

Since $ba, ab \in \mathcal{A}^\bullet$, then by Lemma 2.2, we have

$$bap = ba, \quad p = pba, \quad ab(1 - q) = (1 - q), \quad (1 - q)ab = ab. \quad (2.2)$$

Then from Eq. (2.1) and Eq. (2.2), we can get $ba = pba = p$ and $ab = (1 - q)ab = 1 - q$. This completes the proof. □

Obviously, for an operator $A \in B(X)$, if $A_{T,S}^{(2)}$ exists, we can set $P = A_{T,S}^{(2)}A$ and $Q = I - AA_{T,S}^{(2)}$, then we have $\text{Ran}(A_{T,S}^{(2)}) = \text{Ran}(P)$ and $\text{Ker}(A_{T,S}^{(2)}) = \text{Ran}(Q)$. Thus the (p, q) -outer generalized inverse is a natural algebraic extension of the generalized inverse of linear operators with prescribed range and kernel. Similar to some characterizations of the outer generalized inverse $A_{T,S}^{(2)}$ about matrix and operators, we present the following statements relative to the (p, q) -outer generalized inverse $a_{p,q}^{(2)}$.

Statement 2.5. Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$. Consider the following four statements:

- (1) $a_{p,q}^{(2)}$ exists;
- (2) There exists $b \in \mathcal{A}$ such that $bab = b$, $R_r(b) = R_r(p)$ and $K_r(b) = R_r(q)$;
- (3) $K_r(a) \cap R_r(p) = \{0\}$ and $\mathcal{A} = aR_r(p) \dot{+} R_r(q)$;
- (4) $K_r(a) \cap R_r(p) = \{0\}$ and $aR_r(p) = R_r(1 - q)$.

If we assume statements (1) in 2.5 holds, i.e., $a_{p,q}^{(2)}$ exists, then we can check easily that the other three statements (2), (3) and (4) in Statements 2.5 will hold. Here we give a proof of $(1) \Rightarrow (4)$.

Proposition 2.6. Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$. If $a_{p,q}^{(2)}$ exists, then $K_r(a) \cap R_r(p) = \{0\}$ and $aR_r(p) = R_r(1 - q)$.

Proof. Suppose that $a_{p,q}^{(2)}$ exists, then from its definition, we have $ba = p$ and $ab = 1 - q$. Let $x \in K_r(a) \cap R_r(p)$, then there exists some $t \in R_r(p)$ such that $x = pt$ and $ax = 0$, it follows that $x = bat = babat = bap t = bax = 0$, i.e., $K_r(a) \cap R_r(p) = \{0\}$.

Let $y \in aR_r(p)$, then $y = aps$ for some $s \in \mathcal{A}$, that is $y = aps = abas = (1 - q)as$, thus $y \in R_r(1 - q)$. For any $x \in R_r(1 - q)$, there is some $t \in R_r(1 - q)$ with $x = (1 - q)t$, then $x = abt = ababt = apbt$ and hence $x \in aR_r(p)$. This completes the proof. \square

It is obvious that $(4) \Rightarrow (3)$ holds, but the following example shows that (3) and (4) are not equivalent in general.

Example 2.7. We also consider the matrix algebra $\mathcal{A} = M_2(\mathbb{C})$, and take the same elements $a, p, q \in \mathcal{A}$ as in Example 2.1. Then

$$\begin{aligned} aR_r(p) &= \left\{ \begin{bmatrix} 0 & 0 \\ s & t \end{bmatrix} \mid s, t \in \mathbb{C} \right\}, & R_r(1_2 - q) &= \left\{ \begin{bmatrix} s & t \\ s & t \end{bmatrix} \mid s, t \in \mathbb{C} \right\}, \\ K_r(a) &= \left\{ \begin{bmatrix} 0 & 0 \\ s & t \end{bmatrix} \mid s, t \in \mathbb{C} \right\}, & R_r(p) &= \left\{ \begin{bmatrix} s & t \\ 0 & 0 \end{bmatrix} \mid s, t \in \mathbb{C} \right\}. \end{aligned}$$

It follows that $aR_r(p) \neq R_r(1 - q)$ when $s \neq t \neq 0$ and $\mathcal{A} = aR_r(p) \dot{+} R_r(q)$.

Therefore, from above arguments, (3) and (4) in Statement 2.5 are not equivalent in general.

But similar to the outer inverse $A_{T,S}^{(2)}$, as described by Eq. (1.1), we can prove (2) and (3) in Statements 2.5 are equivalent.

Theorem 2.8. Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$. Then the following statements are equivalent:

- (1) There exists $b \in \mathcal{A}$ such that $bab = b$, $R_r(b) = R_r(p)$ and $K_r(b) = R_r(q)$,
- (2) $K_r(a) \cap R_r(p) = \{0\}$ and $\mathcal{A} = aR_r(p) \dot{+} R_r(q)$.

Proof. (1) \Rightarrow (2) Suppose that there exists $b \in \mathcal{A}$ satisfying (1). Since $bab = b$, then ab is idempotent. Thus we have $\mathcal{A} = R_r(ab) \dot{+} K_r(ab)$. It is easy to check that

$$R_r(ab) = aR_r(b) = aR_r(p) \text{ and } K_r(ab) = K_r(b) = R_r(q).$$

Hence $\mathcal{A} = aR_r(b) \dot{+} R_r(q)$.

Next we show that $K_r(a) \cap R_r(p) = \{0\}$. Let $x \in K_r(a) \cap R_r(p)$. Since $R_r(p) = R_r(b)$, then there exists some $y \in \mathcal{A}$ such that $x = by$ and $ax = 0$. So $x = by = baby = bax = 0$. Therefore, we have $K_r(a) \cap R_r(p) = \{0\}$.

(2) \Rightarrow (1) Suppose that (2) is true. From $\mathcal{A} = aR_r(p) \dot{+} R_r(q)$ and \mathcal{A} is a Banach algebra, we see that $aR_r(p) = R_r(ap)$ is closed in \mathcal{A} . Let $L_a : \mathcal{A} \rightarrow \mathcal{A}$ be the left multiplier on \mathcal{A} , i.e., $L_a(x) = ax$ for all $x \in \mathcal{A}$. Let ϕ be the restriction of L_a on $R_r(p)$, Since $\|L_a\| \leq \|a\|$, we have $\phi \in B(R_r(p), aR_r(p))$ with $\text{Ker}(\phi) = \{0\}$ and $\text{Ran}(\phi) = aR_r(p)$. Thus $\phi^{-1} : aR_r(p) \rightarrow R_r(p)$ is bounded. Since for any $x \in R_r(p)$ and $z \in \mathcal{A}$, $xz \in R_r(p)$, it follows that $\phi(xz) = \phi(x)z$, $\forall x \in R_r(p)$ and $z \in \mathcal{A}$, and then $\phi^{-1}(yz) = \phi^{-1}(y)z$ for any $y \in aR_r(p)$ and $z \in \mathcal{A}$.

Let $Q : \mathcal{A} \rightarrow aR_r(p)$ be the bounded idempotent mapping. Since

$$\mathcal{A} = aR_r(p) \dot{+} R_r(q), R_r(q)\mathcal{A} \subset R_r(q) \text{ and } aR_r(p)\mathcal{A} \subset aR_r(p),$$

it follows that for any $x, z \in \mathcal{A}$, $x = x_1 + x_2$ and $xz = x'_1 + x_2z$, where $x'_1 = x_1z \in aR_r(p)$ and $x_2 \in R_r(q)$, and hence $Q(xz) = Q(x)z$. Set $W = \phi^{-1} \circ Q$. Then

$$W(xz) = W(x)z, \quad (WL_aW)(x) = W(x), \quad \forall x, z \in \mathcal{A}.$$

Put $b = W(1)$. Then from the above arguments, we get that $bab = b$. Since $W(x) = W(1)x$ for any $x \in \mathcal{A}$, we have

$$\begin{aligned} R_r(b) &= R_r(W(1)) = \text{Ran}(W) = R_r(p), \\ K_r(b) &= K_r(W(1)) = \text{Ker}(W) = R_r(q). \end{aligned}$$

Thus (1) holds. □

Now for four statements in Statement 2.5, we have (1) \Rightarrow (4) \Rightarrow (3) \Leftrightarrow (2), and in general, (3) \nRightarrow (4). The following example also shows that statements (1) and (4) in Statement 2.5 are not equivalent in general.

Example 2.9. Let $\mathcal{A} = M_2(\mathbb{C})$ and let $a = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $p = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $1_2 - q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $aR_r(p) = R_r(1_2 - q)$ and $K_r(a) \cap R_r(p) = \{0\}$, i.e., Statement (4) holds.

If $a_{p,q}^{(2)}$ exists, that is, there is $b \in \mathcal{A}$ such that $bab = b$, $ba = p$ and $1_2 - ab = q$. But we could not find $b \in \mathcal{A}$ such that $ba = p$. Thus (4) \nRightarrow (1).

Based on above arguments, we give the following new definition with respect to the outer generalized inverse with prescribed idempotents in a general Banach algebra.

Definition 2.10. Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$. An element $c \in \mathcal{A}$ satisfying

$$cac = c, \quad R_r(c) = R_r(p), \quad K_r(c) = R_r(q)$$

will be called the (p, q, l) -outer generalized inverse of a , written as $a_{p,q}^{(2,l)} = c$.

Remark 2.11. For $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$. Let $L_a: \mathcal{A} \rightarrow \mathcal{A}$ be the left multiplier on \mathcal{A} . If we set $T = R_r(p)$ and $S = R_r(q)$. Then it is obvious that $a_{p,q}^{(2,l)}$ exists in \mathcal{A} if and only if $(L_a)_{T,S}^{(2)}$ exists in the Banach algebra $B(\mathcal{A})$. This shows that why we use the letter “ l ” in Definition 2.10.

Example 2.12. Let $\mathcal{A} = M_2(\mathbb{C})$ and $a, p, q \in \mathcal{A}$ be as in Example 2.1. Simple computation shows that $a_{p,q}^{(2,l)} = b$.

By Lemma 2.2 and Definition 2.10, we have

Corollary 2.13. Suppose that $a, w \in \mathcal{A}$ and $a_{p,q}^{(2,l)}$ exists. Then

- (1) $a_{p,q}^{(2,l)} aw = w$ if and only if $R_r(w) \subset R_r(p)$,
- (2) $wa a_{p,q}^{(2,l)} = w$ if and only if $R_r(q) \subset K_r(w)$.

From Theorem 2.3 and Theorem 2.8, we know that if $a_{p,q}^{(2,l)}$ exists, then it is unique, and the properties of $a_{p,q}^{(2,l)}$ are much more similar to $A_{T,S}^{(2)}$ than $a_{p,q}^{(2)}$. Thus the outer generalized inverse $a_{p,q}^{(2,l)}$ is also a natural extension of the generalized inverses $A_{T,S}^{(2)}$. Also from Definition 2.10, we can see that if $a_{p,q}^{(2,l)} = c$ exists, then we also have $R_r(ca) = R_r(c) = R_r(p)$ and $K_r(ac) = K_r(c) = R_r(q)$.

3 Characterizations for the generalized inverses with prescribed idempotents

By using idempotent elements, firstly, we give the following new characterization of generalized invertible elements in a Banach algebra.

Proposition 3.1 ([16, Theorem 2.4.4]). Let $a \in \mathcal{A}$. Then $a \in Gi(\mathcal{A})$ if and only if there exist $p, q \in \mathcal{A}^\bullet$ such that $K_r(a) = K_r(p)$ and $R_r(a) = R_r(q)$.

Proof. Suppose that $a \in Gi(\mathcal{A})$. Put $p = a^+a$ and $q = aa^+$. Then $p, q \in \mathcal{A}$. Let $x \in K_r(a)$. Then $ax = 0$ and $px = a^+ax = 0$, that is $x \in K_r(p)$. On the other hand, let $y \in K_r(p)$, then $ay = aa^+ax = apy = 0$. So we have $K_r(a) = K_r(p)$.

Similarly, we can check $R_r(a) = R_r(q)$.

Now assume that $K_r(a) = K_r(p)$ and $R_r(a) = R_r(q)$ for $p, q \in \mathcal{A}^\bullet$. Then

$$\mathcal{A} = R_r(p) \dot{+} K_r(p) = R_r(p) \dot{+} K_r(a).$$

Let L be the restriction of L_a on $R_r(p)$. Then L is a mapping from $R_r(p)$ to $R_r(a)$ with $\text{Ker}(L) = \{0\}$ and $\text{Ran}(L) = R_r(a) = R_r(q)$. Thus $L^{-1}: R_r(a) \rightarrow R_r(p)$ is well-defined. Note that $xt \in R_r(p)$ for any $x \in R_r(p)$ and $t \in \mathcal{A}$. So we have

$$L(xt) = L_a(xt) = axt = L_a(x)t = L(x)t$$

and hence $L^{-1}(st) = L^{-1}(s)t$ for any $s \in R_r(a)$ and $t \in \mathcal{A}$.

Put $G = L^{-1} \circ L_q$. Since $q \in \mathcal{A}^\bullet$, we have $L_q: \mathcal{A} \rightarrow R_r(q)$ is an idempotent mapping. Put $b = G(1)$. It is easy to check that $aba = a$ and $bab = b$, i.e., $a \in Gi(\mathcal{A})$. \square

We now present the equivalent conditions about the existence of $a_{p,q}^{(2,l)}$ as follows.

Theorem 3.2. *Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$. Then the following statements are equivalent:*

- (1) $a_{p,q}^{(2,l)}$ exists,
- (2) There exists $b \in \mathcal{A}$ such that $bab = b$, $R_r(b) = R_r(p)$ and $K_r(b) = R_r(q)$,
- (3) $K_r(a) \cap R_r(p) = \{0\}$ and $\mathcal{A} = aR_r(p) \dot{+} R_r(q)$.
- (4) There is some $b \in \mathcal{A}$ satisfying $b = pb$, $p = bap$, $b(1-q) = b$, $1-q = (1-q)ab$.
- (5) $p \in R_l((1-q)ap) = \{x(1-q)ap \mid x \in \mathcal{A}\}$ and $1-q \in R_r((1-q)ap)$,
- (6) There exist some $s, t \in \mathcal{A}$ such that $p = t(1-q)ap$, $1-q = (1-q)aps$.

Proof. (1) \Leftrightarrow (2) comes from the definition of $a_{p,q}^{(2,l)}$. The implication (2) \Leftrightarrow (3) is Theorem 2.8 and the implication (3) \Leftrightarrow (4) is Proposition 2.3. The implication (5) \Rightarrow (6) is obvious.

(1) \Rightarrow (5) Choose $x = b$, then $p = bap = b(1-q)ap \in R_l((1-q)ap)$. Since $1-q = (1-q)ab = (1-q)abp$, then, $1-q \in R_r((1-q)ap)$.

(6) \Rightarrow (4) If $p = t(1-q)ap$ and $1-q = (1-q)aps$ for some $s, t \in \mathcal{A}$, then $t(1-q) = ps$. Set $b = t(1-q) = ps$. Then $pb = pps = b$, $bap = t(1-q)ap = p$ and $b(1-q) = t(1-q)(1-q) = b$, $(1-q)ab = (1-q)aps = 1-q$. \square

If the generalized inverse $a_{p,q}^{(2,l)}$ satisfies $aa_{p,q}^{(2,l)}a = a$, then we call it the $\{1, 2\}$ generalized inverse of $a \in \mathcal{A}$ with prescribed idempotents p and q . It is denoted by $a_{p,q}^{(l)}$. Obviously, $a_{p,q}^{(l)}$ is unique if it exists. The following theorem gives some equivalent conditions about the existence of $a_{p,q}^{(l)}$.

Theorem 3.3. *Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$. Then the following conditions are equivalent:*

- (1) $a_{p,q}^{(l)}$ exists, i.e., there exists some $b \in \mathcal{A}$ such that

$$aba = a, \quad bab = b, \quad R_r(b) = R_r(p), \quad K_r(b) = R_r(q),$$

- (2) $\mathcal{A} = R_r(a) \dot{+} R_r(q) = K_r(a) \dot{+} R_r(p)$,
- (3) $\mathcal{A} = aR_r(p) \dot{+} R_r(q)$, $R_r(a) \cap R_r(q) = \{0\}$, $K_r(a) \cap R_r(p) = \{0\}$.

Proof. (1) \Rightarrow (2) From (1), we have that $(ab)^2 = ab$ and $(ba)^2 = ba$. Also we have the following relation:

$$\begin{aligned} K_r(b) &\subset K_r(ab) \subset K_r(bab) = K_r(b), \quad K_r(ba) \subset K_r(aba) = K_r(a) \subset K_r(ba), \\ R_r(ba) &\subset R_r(b) = R_r(bab) \subset R_r(ba), \quad R_r(ab) \subset R_r(a) = R_r(aba) \subset R_r(ab). \end{aligned}$$

Thus we have

$$\begin{aligned} R_r(ba) &= R_r(b) = R_r(p), \quad R_r(ab) = R_r(a), \\ K_r(ab) &= R_r(b) = R_r(q), \quad K_r(ba) = K_r(a). \end{aligned}$$

From above equations, we can get

$$\mathcal{A} = R_r(a) \dot{+} R_r(q) = K_r(a) \dot{+} R_r(p).$$

(2) \Rightarrow (3) If $\mathcal{A} = R_r(a) \dot{+} R_r(q) = K_r(a) \dot{+} R_r(p)$, then it is obvious that

$$R_r(a) \cap R_r(q) = \{0\}, \quad K_r(a) \cap R_r(p) = \{0\}.$$

So we need only to check that $aK_r(p) = R_r(a)$.

Obviously, we have $aR_r(p) \subset R_r(a)$. Now for any $x \in R_r(a)$, we have $x = at$ for some $t \in \mathcal{A}$. Since $\mathcal{A} = K_r(a) \dot{+} R_r(p)$, we can write $t = t_1 + t_2$, where $t_1 \in K_r(a)$ and $t_2 \in R_r(p)$. Thus $x = at = at_2 \in aR_r(p)$. Therefore $R_r(a) \subset aR_r(p)$ and $R_r(a) = aR_r(p)$.

(3) \Rightarrow (1) Suppose that (3) is true, then by Theorem 3.2, we know that $a_{p,q}^{(2,l)}$ exists, and $R_r(b) = R_r(p)$, $K_r(b) = R_r(q)$. We need to show $aba = a$. Since $bab = b$, then $b(aba - a) = 0$. it follows that

$$aba - a \subset R_r(a) \cap K_r(b) = R_r(a) \cap R_r(q) = \{0\}.$$

i.e., $aba = a$. This completes the proof. \square

The following proposition gives a characterization of $a_{p,q}^{(1,2)}$ in a Banach algebra \mathcal{A} .

Proposition 3.4. *Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$. Then the following statements are equivalent:*

- (1) $a_{p,q}^{(1,2)}$ exists,
- (2) There exists some $b \in \mathcal{A}$ satisfying

$$\begin{cases} aba = a \\ bab = b \end{cases} \quad \text{and} \quad \begin{cases} R_r(b) = R_r(p), \quad K_r(b) = R_r(q) \\ R_r(a) = R_r(1 - q), \quad K_r(a) = R_r(1 - p) \end{cases}.$$

Proof. (1) \Rightarrow (2) This is obvious by the Definition of $a_{p,q}^{(1,2)}$.

(2) \Rightarrow (1) Since $aba = a$ and $bab = b$, so by (2) we have

$$\begin{cases} R_r(ba) = R_r(b) = R_r(p), \quad K_r(ab) = K_r(b) = R_r(q) \\ R_r(ab) = R_r(a) = R_r(1 - q), \quad K_r(ba) = K_r(a) = R_r(1 - p) \end{cases}.$$

Then by Lemma 2.2 and Theorem 3.2, we have

$$b = pb, \quad (1 - q) = (1 - q)ab, \quad a = (1 - q)a, \quad p = pba.$$

Thus, we get

$$ba = pba = p, \quad ab = (1 - q)ab = 1 - q.$$

Since $a_{p,q}^{(1,2)}$ is unique, it follows that $a_{p,q}^{(1,2)} = b$. \square

4 Some representations for the generalized inverses with prescribed idempotents

Let $a \in \mathcal{A}$. If for some positive integer k , there exists an element $b \in \mathcal{A}$ such that

$$(1^k) \quad a^{k+1}b = a^k, \quad (2) \quad bab = b, \quad (5) \quad ab = ba.$$

Then a is Drazin invertible and b is called the Drazin inverse of a , denoted by a^D (cf. [1, 3]). The least integer k is the index of a , denoted by $\text{ind}(a)$. When $\text{ind}(a) = 1$, a^D is called the group inverse of a , denoted by $a^\#$. It is well-known that if the Drazin (group) inverse of a exists, then it is unique. Let \mathcal{A}^D (resp. \mathcal{A}^g) denote the set of all Drazin (resp. group) invertible elements in \mathcal{A} .

The representations of $A_{T,S}^{(2)}$ of a matrix or an operator have been extensively studied. We know that if $A_{T,S}^{(2)}$ exists, then it can be explicitly expressed by the group inverse of AG or GA for some matrix or an operator G with $\text{Ran}(G) = T$ and $\text{Ker}(G) = S$ (cf. [13, 14, 15, 18]). By using the left multiplier representation, we give an explicit representation for the $a_{p,q}^{(2,l)}$ by the group inverse as follows.

Theorem 4.1. *Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$ such that $a_{p,q}^{(2,l)}$ exists. Let $w \in \mathcal{A}$ such that $R_r(w) = R_r(p)$ and $K_r(w) = R_r(q)$. Then $wa, aw \in \mathcal{A}^g$ and $a_{p,q}^{(2,l)} = (wa)^\#w = w(aw)^\#$.*

Proof. Firstly, we show that $a_{p,q}^{(2,l)} = w(aw)^\#$.

Obviously, $R_r(aw) = aR_r(w) = aR_r(p)$. For any $y \in K_r(aw)$, $awy = 0$ and

$$wy \in K_r(a) \cap R_r(w) = K_r(a) \cap R_r(p) = \{0\}.$$

It follows that $wy = 0$ and $y \in K_r(w)$. Thus $K_r(aw) \subset K_r(w)$ and consequently $K_r(aw) = K_r(w) = R_r(q)$. Therefore, $\mathcal{A} = R_r(aw) \dot{+} K_r(aw)$. Set $L = L_{aw}|_{R_r(aw)}$. Then $\text{Ker}(L) = \{0\}$ and $\text{Ran}(L) = R_r(aw)$. Let $P: \mathcal{A} \rightarrow R_r(aw)$ be a projection (idempotent operator). Then $P \in B(\mathcal{A})$ and $L^{-1} \in B(R_r(aw))$. Put $G = L^{-1} \circ P \in B(\mathcal{A})$. Then we have $G(xy) = G(x)y$, $\forall x, y \in \mathcal{A}$ and

$$GL_{aw}G = G, \quad L_{aw}GL_{aw} = L_{aw}, \quad L_{aw}G = GL_{aw}.$$

Put $c = G(1)$. Then $(aw)c(aw) = aw$, $c(aw)c = c$ and $c(aw) = (aw)c$, i.e., $(aw)^\# = c$. So $R_r(aw(aw)^\#) = R_r(aw) = aR_r(p)$. Put $b = w(aw)^\#$. Then

$$bab = w(aw)^\#aw(aw)^\# = w(aw)^\# = b$$

and $R_r(b) = R_r(w(aw)^\#) \subset R_r(w) = R_r(p)$. On the other hand, since

$$K_r((aw)^\#aw) = K_r(aw(aw)^\#) = K_r(aw) = R_r(q) = K_r(w),$$

it follows from Lemma 2.2 that $w(aw)^\#aw = w$. Thus

$$R_r(p) = R_r(w) = R_r(w(aw)^\#aw) \subset R_r(b) \subset R_r(p).$$

Similarly, we have

$$\begin{aligned} K_r(b) &= K_r(w(aw)^\#) \subset K_r(aw(aw)^\#) = K_r(aw) = R_r(q), \\ R_r(q) &= K_r(w) = K_r(aw(aw)^\#) \subset K_r(w(aw)^\#aw(aw)^\#) \\ &= K_r(w(aw)^\#) = K_r(b). \end{aligned}$$

Thus, $K_r(b) = R_r(q)$. So by the unique of $a_{p,q}^{(2,l)}$, we have $b = w(aw)^\# = a_{p,q}^{(2,l)}$.

Similarly, if we put $d = (wa)^\#w$, then we can prove $d = a_{p,q}^{(2,l)} = b$. \square

Corollary 4.2. *Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$ such that $a_{p,q}^{(2,l)}$ exists. Let $w \in \mathcal{A}$ with $R_r(w) = R_r(p)$ and $K_r(w) = R_r(q)$. Then there exists some $c \in \mathcal{A}$ such that*

$$wawc = w \quad \text{and} \quad a_{p,q}^{(2,l)}awc = a_{p,q}^{(2,l)}. \quad (4.1)$$

Proof. From Theorem 4.1 above, we know that $(aw)^\#$ exists. Put $c = (aw)^\#$. Then by Lemma 2.2 and Theorem 4.1, we see that

$$K_r(a_{p,q}^{(2,l)}) = R_r(q) = K_r(w) = K_r(aw) = K_r(awc).$$

Thus $c = (aw)^\#$ satisfies Eq.(4.1). \square

Similar to Theorem 4.1, we have the following easy representation of $a_{p,q}^{(2)}$.

Theorem 4.3. *Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$ such that $a_{p,q}^{(2)}$ exists. Let $w \in \mathcal{A}$ such that $wa = p$ and $aw = 1 - q$. Then $a_{p,q}^{(2)} = (wa)^\#w = w(aw)^\#$.*

Proof. Obviously, $wa, aw \in \mathcal{A}^g$ for $wa = p$ and $aw = 1 - q$. We also have $(wa)^\# = p$ and $(aw)^\# = 1 - q$. Then by using the uniqueness of $a_{p,q}^{(2)}$, we can prove this theorem by simple computation. \square

From the Definition of $a_{p,q}^{(2)}$ and $a_{p,q}^{(2,l)}$, it is easy to see that if $a_{p,q}^{(2)}$ exists, then $a_{p,q}^{(2,l)}$ exists. In this case, $a_{p,q}^{(2)} = a_{p,q}^{(2,l)} = b$ by the uniqueness. Thus, Theorem 4.1 and Corollary 4.2 also hold if we replace $a_{p,q}^{(2,l)}$ by $a_{p,q}^{(2)}$. The following result also gives some generalizations of [4, Theorem 2.2] and [5, Theorem 1.2].

Corollary 4.4. *Let $a, c \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$ such that $a_{p,q}^{(2,l)}$ and $c_{1-q,1-p}^{(1,2)}$ exist. Then ac and ca is group invertible and $a_{p,q}^{(2,l)} = (ca)^\#c = c(ac)^\#$.*

Proof. Since $c_{1-q,1-p}^{(1,2)}$ exists, then by Proposition 3.4, we have $R_r(c) = R_r(p)$, $K_r(c) = R_r(q)$. Thus, we can get our results by using Theorem 4.1. \square

In the following theorem, we give a simple representation of $a_{p,q}^{(2,l)}$ based on $\{1\}$ inverse. The analogous results about outer generalized inverse with prescribed range and null space of Banach space operators were presented in [20, Theorem 2.1].

Theorem 4.5. *Let $p, q \in \mathcal{A}^\bullet$. Let $a, w \in \mathcal{A}$ with $R_r(w) = R_r(p)$ and $K_r(w) = R_r(q)$. Then the following conditions are equivalent:*

- (1) $a_{p,q}^{(2,l)}$ exists;
- (2) $(aw)^\#$ exists and $K_r(a) \cap R_r(w) = \{0\}$;
- (3) $(wa)^\#$ exists and $R_r(w) = R_r(waw)$.

In this case, waw is regular and

$$a_{p,q}^{(2,l)} = (wa)^\# w = w(aw)^\# = w(waw)^- w. \quad (4.2)$$

Proof. (1) \Rightarrow (2) follows from Theorem 3.2 and Theorem 4.1.

(2) \Rightarrow (3) Since $(aw)^\#$ exists, we have $a(waw(aw)^\# - w) = 0$. So

$$waw(aw)^\# - w \in K_r(a) \cap R_r(w) = \{0\},$$

i.e., $waw(aw)^\# = w$. Therefore $R_r(w) = R_r(waw)$.

Let $c = w((aw)^\#)^2 a$, we show that c is the group inverse of wa . In fact,

$$\begin{aligned} c(wa)c &= w((aw)^\#)^2 (aw)(aw)((aw)^\#)^2 a = w(aw)^\# (aw)^\# a = c, \\ (wa)c(wa) &= waw((aw)^\#)^2 awa = (waw(aw)^\#)a = wa, \\ (wa)c &= waw((aw)^\#)^2 a = w(((aw)^\#)^2 aw)a = c(wa). \end{aligned}$$

i.e., $(wa)^\#$ exists and $c = (wa)^\#$.

(3) \Rightarrow (1) Since $R_r(w) = R_r(waw)$, $R_r(wa) \subset R_r(w) = R_r(waw) \subset R_r(wa)$. So $R_r(w) = R_r(wa)$. The existence of $(wa)^\#$ means that

$$K_r(wa) \cap R_r(wa) = \{0\} \text{ and } wa(wa(wa)^\# w - w) = 0.$$

So

$$wa(wa)^\# w - w \in K_r(wa) \cap R_r(w) = K_r(wa) \cap R_r(wa) = \{0\}.$$

Hence $w = wa(wa)^\# w$. Set $b = (wa)^\# w$, then we have

$$\begin{aligned} bab &= (wa)^\# wa(wa)^\# w = (aw)^\# w = b, \\ R_r(p) &= R_r(w) = R_r(wa(wa)^\# w) = R_r((wa)^\# waw) \subset R_r((wa)^\# w) \\ &= R_r(b) = R_r((wa)^\# wa(wa)^\# w) \\ &= R_r(wa((wa)^\#)^2 w) \subset R_r(w) \\ &= R_r(p), \\ R_r(q) &= K_r(w) \subset K_r((wa)^\# w) = K_r(b) \subset K_r(wa(wa)^\# w) = K_r(w) \\ &= R_r(q). \end{aligned}$$

Thus, $R_r(b) = R_r(p)$ and $K_r(b) = R_r(q)$ and consequently, by Theorem 3.2, we see $a_{p,q}^{(2,l)}$ exists and $a_{p,q}^{(2,l)} = (wa)^\# w$.

Now we show that waw is regular and Eq. (4.2) is true when one of (1), (2) or (3) in the Theorem 4.5 holds. Since

$$R_r((wa)^\# wa) = R_r(wa) = R_r(w) \supset R_r(w(aw)^\#),$$

therefore, by Lemma 2.2, we have $((wa)^\# wa)w(aw)^\# = w(aw)^\#$. Then

$$\begin{aligned} w(aw)^\# &= ((wa)^\# wa)w(aw)^\# = ((wa)^\# wa)^2 w(aw)^\# \\ &= ((wa)^\#)^2 waw = (wa)^\# w. \end{aligned}$$

Thus we have $a_{p,q}^{(2,l)} = w(aw)^\#$.

Set $x = a(wa)^\#^2$. Then it is easy to check that

$$(waw)x(waw) = (waw)a(wa)^\#^2(waw) = waw.$$

i.e., waw is regular and $(waw)^- = a(wa)^\#^2$. Furthermore, we have the following,

$$w(waw)^- w = wa(wa)^\#^2 w = (wa)^\# w = a_{p,q}^{(2,l)}.$$

This completes the proof. \square

In paper [5], a representation of the (p, q) -outer generalized inverse based on $(1, 5)$ inverse over Banach algebra is presented. But as in our introduction in this paper, we indicate that Theorem 1.4 of [5] is wrong. We have to say that Theorem 2.1 in [5] is also wrong since the proof mainly based on [5, Theorem 1.4]. In fact, Theorem 2.1 in [5] gives the representation of $a_{p,q}^{(2,l)}$, not $a_{p,q}^{(2)}$. The following is a corrected (and generalized) version of their theorem. This result is also an improvement of Theorem 4.5, which removes the existence of the group inverses of wa or aw .

Theorem 4.6. *Let $a, w \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$ such that $w_{1-q, 1-p}^{(1,2)}$ exist (or $R_r(w) = R_r(p)$ and $K_r(w) = R_r(q)$). Then the following statements are equivalent:*

- (1) $a_{p,q}^{(2,l)}$ exists;
- (2) $(aw)^{(1,5)}$ exists and $K_r(a) \cap R_r(w) = \{0\}$;
- (3) $(wa)^{(1,5)}$ exists and $R_r(w) = R_r(wa)$.

In this case, waw is inner regular and

$$a_{p,q}^{(2,l)} = (wa)^{(1,5)} w = w(aw)^{(1,5)} = w(waw)^- w.$$

Proof. We only need to follow the line of proof Theorem 2.1 in [5], but make some essential modifications by using Definition 2.10 and Theorem 3.2. Here we omit the detail. \square

Based on an explicit representation for $a_{p,q}^{(2,l)}$ by the group inverse, now we can give some limit and integral representations of the (p, q, l) -outer generalized inverse $a_{p,q}^{(2,l)}$, the analogous result is well-known for operators on Banach space (see [16]).

Theorem 4.7. *Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$ such that $a_{p,q}^{(2,l)}$ exists. Let $w \in \mathcal{A}$ such that $R_r(w) = R_r(p)$ and $K_r(w) = R_r(q)$. Then $a_{p,q}^{(2,l)} = \lim_{\substack{\lambda \rightarrow 0 \\ \lambda \notin \sigma(-aw)}} w(\lambda 1 + aw)^{-1}$.*

Proof. By Theorem 4.1, we know that $aw \in \mathcal{A}^g$ and $a_{p,q}^{(2,l)} = w(aw)^\#$. Let $f = aa_{p,q}^{(2,l)}$. Then $f \in \mathcal{A}^\bullet$ and $f = aw(aw)^\#$. Also from the proof of Theorem 4.1, we see that $aw, (aw)^\# \in f\mathcal{A}f$ and aw is invertible in $f\mathcal{A}f$ with $(aw|_{f\mathcal{A}f})^{-1} = (aw)^\#$. Consider the decomposition $\mathcal{A} = f\mathcal{A}f \oplus (1-f)\mathcal{A}(1-f)$. Then we can write $\lambda 1 + aw$ as the following matrix form

$$\lambda 1 + aw = \begin{bmatrix} \lambda f + aw & \\ & \lambda(1-f) \end{bmatrix} \begin{matrix} f\mathcal{A}f \\ (1-f)\mathcal{A}(1-f) \end{matrix} \quad (4.3)$$

It is well-known that if $\lambda \notin \sigma(-aw) \cup \{0\}$, then $\lambda f + aw$ is invertible in $f\mathcal{A}f$. Thus, in the case, by Eq.(4.3) we have

$$(\lambda 1 + aw)^{-1} = \begin{bmatrix} (\lambda f + aw)^{-1} & \\ & \lambda^{-1}(1-f) \end{bmatrix} \begin{matrix} f\mathcal{A}f \\ (1-f)\mathcal{A}(1-f) \end{matrix} \quad (4.4)$$

Since $K_r(w) = R_r(q)$, then from the proof of Theorem 4.1, we see that $K_r(w) = R_r(q) = K_r(aw(aw)^\#) = K_r(f) = R_r(1-f)$. So by Eq. (4.4) we have $w(\lambda 1 + aw)^{-1} = w(\lambda 1 + aw)^{-1}f$, where the inverse is taken in $f\mathcal{A}f$. Then we can compute in the following way

$$a_{p,q}^{(2,l)} = w(aw)^\#f = \lim_{\substack{\lambda \rightarrow 0 \\ \lambda \notin \sigma(-aw)}} w(\lambda 1 + aw)^{-1}f = \lim_{\substack{\lambda \rightarrow 0 \\ \lambda \notin \sigma(-aw)}} w(\lambda 1 + aw)^{-1}.$$

This completes the proof. \square

We need the following lemma in a Banach algebra.

Lemma 4.8 ([16, Proposition 1.4.17]). *Let $a \in \mathcal{A}$. Suppose that $\operatorname{Re}(\lambda) < 0$ for every $\lambda \in \sigma(a)$. Then*

$$a^{-1} = \int_0^{+\infty} e^{at} dt \triangleq \lim_{x \rightarrow +\infty} \int_0^x e^{at} dt.$$

Theorem 4.9. *Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$ such that $a_{p,q}^{(2,l)}$ exists. Let $w \in \mathcal{A}$ such that $R_r(w) = R_r(p)$ and $K_r(w) = R_r(q)$. Suppose that $\operatorname{Re}(\lambda) \geq 0$ for every $\lambda \in \sigma(aw)$. Then*

$$a_{p,q}^{(2,l)} = \int_0^{+\infty} we^{-(aw)t} dt.$$

Proof. We follow the method of proof [16, Corollary 4.2.11], but by using Theorem 4.1 above. It follows from Theorem 4.1 that $aw \in \mathcal{A}^g$ and $a_{p,q}^{(2,l)} = w(aw)^\#$. Similar to Theorem 4.7, Put $g = aw(aw)^\#$. Then $g \in \mathcal{A}^\bullet$ and aw is invertible in $g\mathcal{A}g$ with $(aw|_{g\mathcal{A}g})^{-1} = (aw)^\#$. Obviously, from the proof of Theorem 4.1, we can write aw and $(aw)^\#$ as the following matrix forms, respectively.

$$aw = \begin{bmatrix} aw & \\ & 0 \end{bmatrix} \begin{matrix} g\mathcal{A}g \\ (1-g)\mathcal{A}(1-g) \end{matrix}, \quad (aw)^\# = \begin{bmatrix} (aw)^{-1} & \\ & 0 \end{bmatrix} \begin{matrix} g\mathcal{A}g \\ (1-g)\mathcal{A}(1-g) \end{matrix}.$$

Since $\operatorname{Re}(\lambda) \geq 0$ for every $\lambda \in \sigma(aw)$, then we have $\operatorname{Re}(\lambda) > 0$ for every $\lambda \in \sigma(aw|_{g\mathcal{A}g})$. Thus by Lemma 4.8, we have $(aw|_{g\mathcal{A}g})^{-1} = \int_0^{+\infty} e^{-(aw)t} dt$. Note that $K_r(w) = R_r(1 - g)$. Thus, we have

$$a_{p,q}^{(2,l)} = w(aw)^{\#} = w \begin{bmatrix} (aw)^{-1} & \\ & 1 - g \end{bmatrix} = \int_0^{+\infty} we^{-(aw)t} dt.$$

This completes the proof. \square

Let $a \in \mathcal{A}$. The element a^d is the generalized Drazin inverse, or Koliha–Drazin inverse of $a \in \mathcal{A}$ (see [7]), provided that the following hold:

$$(1^\infty) \quad a(1 - a^d) \text{ is quasinilpotent}, \quad (2) \quad a^d a a^d = a^d, \quad (5) \quad a a^d = d^d a.$$

It is known that $a \in \mathcal{A}$ is generalized Drazin invertible if and only if 0 is not the point of accumulation of the spectrum of a (see [7, 16]).

In the case when \mathcal{A} is a unital C^* -algebra, then the Moore–Penrose inverse of $a \in \mathcal{A}$ (see [8, 12]) is the unique $a^\dagger \in \mathcal{A}$ (in the case when it exists), such that the following hold:

$$(1) \quad a a^\dagger a = a, \quad (2) \quad a^\dagger a a^\dagger = a^\dagger, \quad (3) \quad (a a^\dagger)^* = a a^\dagger, \quad (4) \quad (a^\dagger a)^* = a^\dagger a.$$

For an element $a \in \mathcal{A}$, if a^\dagger exists, then a is called Moore–Penrose invertible. The set of all $a \in \mathcal{A}$ that possess the Moore–Penrose inverse will be denoted by \mathcal{A}^\dagger . It is well-known that for $a \in \mathcal{A}$, a^\dagger exists if and only if a is inner regular (see [6, 9]). The following corollary shows that for any $a \in \mathcal{A}$, a^\dagger , a^d and $a^\#$, if they exist, are all the special cases of $a_{p,q}^{(2)}$, in this case, we have $a_{p,q}^{(2)} = a_{p,q}^{(2,l)}$.

Corollary 4.10. *Let \mathcal{A} be a unital C^* -algebra and $a \in \mathcal{A}$.*

- (1) *If $a \in \mathcal{A}^d$ and $p = 1 - q = 1 - a^\pi = 1 - a a^d = 1 - a^d a$ is the spectral idempotent of a , then $a^d = a_{1-p,p}^{(2)} = a_{q,1-q}^{(2,l)} = a_{q,1-q}^{(2)}$.*
- (2) *If $a \in \mathcal{A}^\dagger$, $p = a^\dagger a$, $q = 1 - a a^\dagger$, then $a^\dagger = a_{p,q}^{(2)} = a_{p,q}^{(2,l)}$.*

Proof. It is routine to check these by the definitions of a^\dagger , a^d , $a_{p,q}^{(2)}$ and $a_{p,q}^{(2,l)}$. \square

In this paper, we give some characterizations and representations for the various different generalized inverses with prescribed idempotents in Banach algebras. Obviously, most of our results can be proved in a ring. But in our forthcoming paper, we will use the main results in this paper to discuss the perturbation analysis of the generalized inverses $a_{p,q}^{(2)}$ and $a_{p,q}^{(2,l)}$ in a Banach algebra.

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